Notes on the correlation between a function and its derivative or first difference

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We demonstrate the absence of correlation between a function satisfying certain weak boundedness conditions and its first derivative. Before attending to the technicalities, we note that the proofs for both theorems are almost immediate from the observation that $\int_a^b x\,\dot{x}\,dt = [\frac{1}{2}x^2]_a^b$.

THEOREM 1 *Let x be a differentiable real function, defined in the interval $[a, b]$, such that $x(a) = x(b)$. If x is not constant then the correlation of x and $\dot{x}$ over $[a, b]$ is defined and equal to zero.*

PROOF. Write $\overline{x}_{a,b}$ and $\overline{\dot{x}}_{a,b}$ for the means of x and $\dot{x}$ over $[a, b]$. By replacing x by $x - \overline{x}_{a,b}$ we may assume without loss of generality that $\overline{x}_{a,b}$ is zero. $\overline{\dot{x}}_{a,b}$ must exist and equal zero, since

$$\overline{\dot{x}}_{a,b} = \frac{1}{b-a}\int_a^b \dot{x}\,dt = \frac{x(b)-x(a)}{b-a} = 0$$

The correlation between x and $\dot{x}$ over $[a, b]$ is defined by:

$$\begin{aligned} c_{x,\dot{x}} &= \frac{\frac{1}{b-a}\int_a^b x\,\dot{x}\,dt}{\sqrt{(\frac{1}{b-a}\int_a^b x^2\,dt)\,(\frac{1}{b-a}\int_a^b \dot{x}^2\,dt)}} \\ &= \frac{(x(b)^2 - x(a)^2)/2}{\sqrt{(\int_a^b x^2\,dt)\,(\int_a^b \dot{x}^2\,dt)}} \end{aligned}$$

The numerator is zero and the denominator is positive (since neither x nor $\dot{x}$ is identically zero). Therefore $c_{x,\dot{x}} = 0$.

THEOREM 2 *Let x be a differentiable real function. Let $\overline{x}$ and $\overline{\dot{x}}$ be the averages of x and $\dot{x}$ over the whole real line. If these averages exist, and if the correlation of x and $\dot{x}$ over the whole real line exists, then the correlation is zero.*

PROOF. Note that the existence of the correlation implies that x is not constant. As before, we can take $\overline{x}$ to be zero and prove that $\overline{\dot{x}}$ is also zero. The correlation between x and $\dot{x}$ is then given by the limit:

$$\begin{aligned} c_{x,\dot{x}} &= \lim_{a \to -\infty,\, b \to \infty} \frac{\frac{1}{b-a} \int_a^b x\, \dot{x}\, dt}{\sqrt{(\frac{1}{b-a} \int_a^b x^2\, dt)\ (\frac{1}{b-a} \int_a^b \dot{x}^2\, dt)}} \\ &= \lim_{a \to -\infty,\, b \to \infty} \frac{(x(b)^2 - x(a)^2)/2}{\sqrt{(\int_a^b x^2\, dt)\ (\int_a^b \dot{x}^2\, dt)}} \end{aligned}$$

Since this limit is assumed to exist, to prove that it is zero it is sufficient to construct some particular sequence of values of a and b tending to $\pm\infty$, along which the limit is zero.

Either $x(b)$ tends to zero as $b \to \infty$, or (since $\overline{x} = 0$ and x is continuous) there are arbitrarily large values of b for which $x(b) = 0$. In either case, for any $\epsilon > 0$ there exist arbitrarily large values of b such that $|x(b)| < \epsilon$. Similarly, there exist arbitrarily large negative values a such that $|x(a)| < \epsilon$. For such a and b, the numerator of the last expression for $c_{x,\dot{x}}$ is less than $\epsilon^2/2$. However, the denominator is positive and non-decreasing as $a \to -\infty$ and $b \to \infty$. The denominator is therefore bounded below for all large enough a and b by some positive value δ.

If we take a sequence ϵ_n tending to zero, and for each ϵ_n take values a_n and b_n as described above, and such that $a_n \to -\infty$ and $b_n \to \infty$, then along this route to the limit, the corresponding approximant to the correlation is less than ϵ_n/δ. This sequence tends to zero, therefore the correlation is zero.

The conditions that $x(a) = x(b)$ in the first theorem and the existence of $\overline{x}$ in the second are essential. If we take $x = e^t$, which violates both conditions, then $\dot{x} = x$ and the correlation is 1 over every finite time interval. That $\overline{\dot{x}}$ and $c_{x,\dot{x}}$ exist is a technicality that rules out certain pathological cases such as $x = \sin(\log(1 + |t|))$, which are unlikely to arise in any practical application.

Both theorems have finite difference versions for time series. They hold for essentially the same reason as the continuous versions: that $(x_i + x_{i+1})(x_{i+1} - x_i) = x_{i+1}^2 - x_i^2$. The proofs are easily obtained from those above.